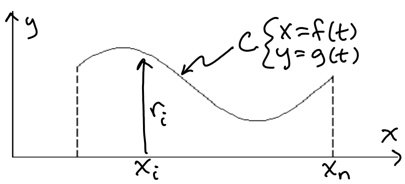
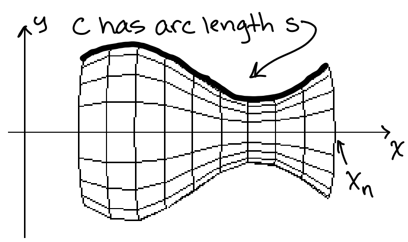
Example: Circumference of a circle $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$
$$= \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = a\int_0^{2\pi} 1\,dt$$
$$= a t\Big|_0^{2\pi} = 2\pi a \checkmark$$

Area of Surface of Revolution

With rectangular equations $y = f(x)$ over $x_0 \le x \le x_1$ revolved about one of the coordinate axes $S = 2\pi\int_a^b r(x)\sqrt{1+(f'(x))^2}\,dx$ where $r(x)$ is the distance between the graph of f and the axis of revolution. This came from the lateral surface area of a strip of the surface: $2\pi r \ell$.





In parametric form: $x = f(t)$, $y = g(t)$, $a \le t \le b$

Revolve about x-axis $g(t) \ge 0$: $S = 2\pi\int_a^b \underset{\text{or } y}{g(t)}\sqrt{(f'(t))^2 + (g'(t))^2}\,dt$
Revolve about y-axis $f(t) \ge 0$: $S = 2\pi\int_a^b \underset{\text{or } f(t)}{x(t)}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Note: Both notations (x-y vs. f-g) apply in both cases.

To help remember this formula, consider the formula for the lateral surface area of a cylinder, $2\pi r h$, that you find by "unrolling" the cylinder. This is the same idea, but r varies and "h" is arc length.

Example: $x = t^3$, $y = t+2$, $1 \le t \le 2$, about y-axis

$$S = 2\pi r \ell$$
$$= 2\pi\int_a^b \overset{x(t)}{x}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$
$$= 2\pi\int_1^2 t^3\sqrt{(3t^2)^2 + (1)^2}\,dt$$
$$= 2\pi\int_1^2 t^3\sqrt{9t^4+1} \qquad \begin{aligned} u &= 9t^4+1 \\ du &= 36t^3\,dt \end{aligned}$$
$$= \frac{2\pi}{36}\int_{10}^{145} u^{1/2}\,du = \frac{\pi}{18}\cdot\frac{2}{3}\left(145^{3/2} - 10^{3/2}\right)$$
$$= \frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)$$