SEQUENCES (page 1 of 4) 11.1 Notes Filled In

Sequence → A function whose domain is the set of positive integers. Informally, it is a list of numbers (or functions) that follow a particular pattern.

Notation: $\{a_n\}$ with $a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$ as terms.

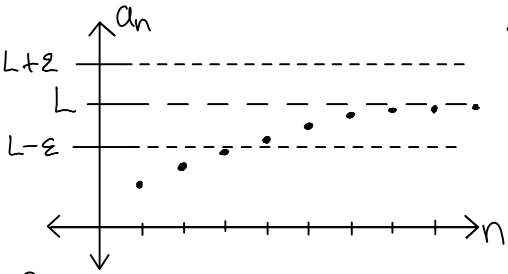
Example: $\left\{\frac{n+1}{n}\right\} \Rightarrow a_n = \frac{n+1}{n} \Rightarrow$ First few terms are:

Note: Sometimes we will start a sequence with the 0^{th} term, a_0.
We can also consider $\{a_n\}$ to be $f(x)$ where $x \in \{1, 2, 3, \ldots\} = \mathbb{Z}^+$
Ex: $f(x) = \frac{x+1}{x}$ with domain $\mathbb{Z}^+$ ← positive integers

If the terms of $\{a_n\}$ approach a limiting value, then the sequence converges.

Ex: $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$ What is going to happen?
This sequence converges to ___.

Formally, $\lim_{n\to\infty} a_n = L$ if for all $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$.
(Shorthand: $\lim_{n\to\infty} a_n = L \Leftrightarrow \forall \varepsilon > 0, \exists M > 0 \ni |a_n - L| < \varepsilon, \forall n > M$.)



This means that if $\{a_n\}$ converges, you can find an M so that $a_M, a_{M+1}, a_{M+2}, \ldots$ etc. are each within ε units of L, no matter how small ε is. (More on this later.)

☆ If f is a function where $f(n) = a_n$, $\forall n \in \mathbb{Z}^+$, then $\lim_{x\to\infty} f(x) = \lim_{n\to\infty} a_n = L$
↑ This is one method of finding the limit of a sequence.

Ex: $\left\{\frac{n+1}{n}\right\} \Rightarrow f(x) = \frac{x+1}{x}$

Note: You cannot differentiate with respect to "n"! a_n is not a continuous function.

$\lim_{x\to\infty} \frac{x+1}{x} \overset{LR}{=} \lim_{x\to\infty} \frac{1}{1} = 1 \Rightarrow \lim_{n\to\infty} \frac{n+1}{n} = 1$

(→ $\frac{\infty}{\infty}$)

Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$

① $\lim_{n\to\infty} (a_n \pm b_n) = L + K$ ② $\lim_{n\to\infty} c a_n = cL$

③ $\lim_{n\to\infty} (a_n b_n) = LK$ ④ $\lim_{n\to\infty} a_n/b_n = L/K$, if $b_n \neq 0$, $K \neq 0$

↑ Same basic limit rules as before ↗

Except L'Hôpital's ⇒ Must make it a function of x first. Otherwise, the derivatives don't exist.

Convergence or Divergence?

Ex: $\{a_n\} = \{3 + (-1)^n\}$

Terms: (n=1) 2, (n=2) 4, (n=3) 2, (n=4) 4, 2, 4, …

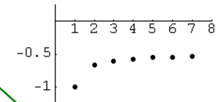
$\lim_{n\to\infty} a_n$ = Does Not Exist (Does not approach one value)

Ex: $\{a_n\} = \left\{\frac{n}{1-2n}\right\}$ Terms: $\frac{1}{-1}, \frac{2}{-3}, \frac{3}{-5}, \frac{4}{-7}, \ldots$

Use $f(x) = \frac{x}{1-2x}$

$\lim_{x\to\infty} \frac{x}{1-2x} \overset{LR}{=} \lim_{x\to\infty} \frac{1}{-2} = -\frac{1}{2}$ ($\to \frac{\infty}{-\infty}$)

$\Rightarrow \lim_{n\to\infty} a_n = -\frac{1}{2}$

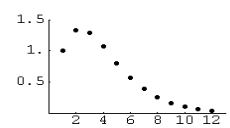


Ex: $\{a_n\} = \left\{\frac{n^2}{2^n - 1}\right\}$, use $f(x) = \frac{x^2}{2^x - 1}$

Terms: $\frac{1}{1}, \frac{4}{3}, \frac{9}{7}, \frac{16}{15}, \frac{25}{31}, \ldots$

$\lim_{x\to\infty} \frac{x^2}{2^x - 1} \overset{LR}{=} \lim_{x\to\infty} \frac{2x}{2^x \ln 2} \overset{LR}{=} \lim_{x\to\infty} \frac{2}{2^x (\ln 2)^2} = 0$ ($\to \frac{\infty}{\infty}$, $\to \frac{\infty}{\infty}$, $\to \frac{1}{\infty}$)

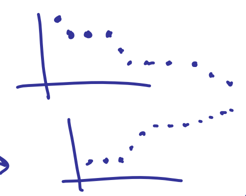
Therefore $\therefore \lim_{n\to\infty} a_n = 0$



A sequence is monotonic if its terms are

nonincreasing ⇒ $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$

or nondecreasing ⇒ $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$



$a_n = 3 + (-1)^n$
Not monotonic

$a_n = \frac{n}{1-2n}$
monotonic increasing
$a_{n+1} > a_n$

$\frac{(n+1)}{1-2(n+1)} \geq \frac{n}{1-2n}$

$n - 2n^2 + 1 - 2n \geq n - 2n^2 - 2n$

$n + 1 \geq n$

$1 \geq 0$ True

$a_n = \frac{n^2}{2^n - 1}$
Not monotonic
$1 < \frac{4}{3}$
$\frac{4}{3} > \frac{25}{31}$

A sequence is bounded above if $\exists M \in \mathbb{R}$ such that $a_n \leq M \ \forall n$ (∃: There Exists; ℝ: Real Numbers; ∀: For all; M: The upper bound)

bounded below if $\exists N \in \mathbb{R}$ such that $N \leq a_n \ \forall n$. (N: The lower bound)

$\{a_n\}$ is bounded if it is bounded above and bounded below.

★ If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

} The converse is not necessarily true.

$a_n = 3 + (-1)^n$

Bounded above by 4
Bounded below by 2
⇒ Sequence is Bounded
but since it is not monotonic we cannot conclude anything about its convergence or divergence based on this info.

$a_n = \frac{n}{1-2n}$

monotonic
Bounded above by $-\frac{1}{2}$
Bounded below by -1
∴ Converges since its Bounded and Monotonic

$a_n = \frac{n^2}{2^n - 1}$

Not monotonic
Bounded above by $\frac{4}{3}$
Bounded below by 0
However, for $n \geq 3$ it is monotonic decreasing
So we can conclude that it converges.

$a_{n+1} \leq a_n$

$\frac{(n+1)^2}{2^{(n+1)} - 1} \leq \frac{n^2}{2^n - 1}$ for $n \geq 3$

★ Squeeze Theorem for Sequences

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ and $\exists N$ such that $a_n \leq c_n \leq b_n \ \forall n > N$, then $\lim_{n\to\infty} c_n = L$

Ex: $\{c_n\} = \{(-1)^n \frac{1}{n!}\}$

Let $a_n = \frac{-1}{2^n}$

$b_n = \frac{1}{2^n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-1}{2^n} = 0 \quad \left(\nearrow \frac{-1}{\infty}\right)$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2^n} = 0 \quad \left(\nearrow \frac{1}{\infty}\right)$

Compare $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \cdots \cdot (n-1) \cdot n$

and $2^n = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2 \cdot 2$

Since $n!$ __>__ 2^n for large n

then $\frac{1}{n!}$ __<__ $\frac{1}{2^n}$

and $\frac{-1}{n!}$ __>__ $\frac{-1}{2^n}$ for large n.

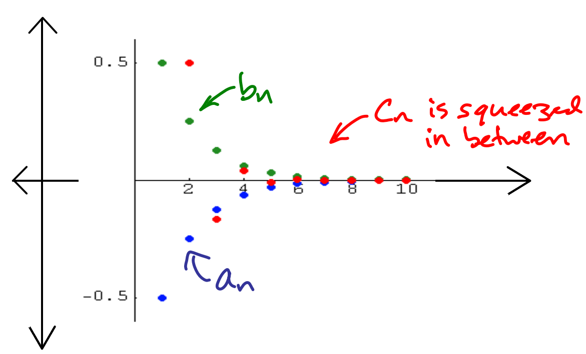
$a_n \leq c_n \leq b_n$

$-\frac{1}{2^n} \leq (-1)^n \frac{1}{n!} \leq \frac{1}{2^n}$ for large n

$\lim_{n\to\infty} -\frac{1}{2^n} \leq \lim_{n\to\infty} (-1)^n \frac{1}{n!} \leq \lim_{n\to\infty} \frac{1}{2^n}$

$0 \leq \lim_{n\to\infty} (-1)^n \frac{1}{n!} \leq 0$

$\Rightarrow \lim_{n\to\infty} (-1)^n \frac{1}{n!} = 0$ also



Note: It can be shown for any k, $\lim_{n\to\infty} \frac{k^n}{n!} = 0$
Factorials go to ∞ faster than any exponential function!

✱ Absolute Value Theorem

For the sequence $\{a_n\}$, if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

(Proof uses the Squeeze Thm.)

Ex: $\{c_n\} = \{(-1)^n \frac{1}{n!}\}$

$$\lim_{n\to\infty} |c_n| = \lim_{n\to\infty} \left|(-1)^n \frac{1}{n!}\right|$$
$$= \lim_{n\to\infty} \frac{1}{n!}$$
$$= 0$$

← This only works for a limit of 0. see example below

Since $\lim_{n\to\infty} |c_n| = 0$, $\lim_{n\to\infty} c_n = 0$ also

$$\Rightarrow \lim_{n\to\infty} (-1)^n \frac{1}{n!} = 0$$

Determining Convergence for Sequences

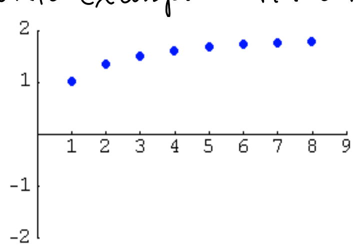
→ Limit Exists
→ Bounded and Monotonic
→ Squeeze Theorem
→ Absolute Value

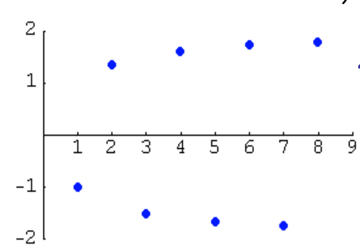
Finding Limit (if exists)

→ Find $\lim_{x\to\infty}$ of corresponding $f(x)$.
↳ You may need L'Hôpital's Rule
↳ You may need to compare $n!$ to exponentials and/or polynomials

→ Absolute Value Theorem
↳ Only if Limit = 0
Counterexample: $\lim_{n\to\infty} \frac{2n}{n+1}$ vs. $\lim_{n\to\infty} (-1)^n \left(\frac{2n}{n+1}\right)$

$\lim_{n\to\infty} \frac{2n}{n+1} = 2$





$\lim_{n\to\infty} (-1)^n \left(\frac{2n}{n+1}\right)$ D.N.E.