Logistic Differential Equation

$P = P(t)$

$$\frac{dP}{dt} = kP\left(1-\frac{P}{K}\right)$$

Separation of variables

$$\frac{1}{P\left(1-\frac{P}{K}\right)}\,dP = k\,dt$$

$$\star \int \frac{1}{P} + \frac{1}{K\left(1-\frac{P}{K}\right)}\,dP = \int k\,dt$$

Partial Fractions

$$\frac{1}{P\left(1-\frac{P}{K}\right)} = \frac{A}{P} + \frac{B}{\left(1-\frac{P}{K}\right)}$$

$$1 = A\left(1-\frac{P}{K}\right) + BP$$

$P = K \Rightarrow 1 = BK \Rightarrow B = \frac{1}{K}$

$P = 0 \Rightarrow 1 = A$

$$\ln|P| + \frac{1}{K}\cdot(-K)\ln\left|1-\frac{P}{K}\right| = kt + C$$

$$\ln|P| - \ln\left|1-\frac{P}{K}\right| = kt + C$$

$$\ln\left|\frac{P}{1-\frac{P}{K}}\right| = kt + C$$

$$\frac{P}{1-\frac{P}{K}} = e^{kt+C}$$

$$P = \left(1-\frac{P}{K}\right)Ce^{kt}$$

$$P = Ce^{kt} - \frac{P}{K}Ce^{kt}$$

$$P\left(1+\frac{1}{K}Ce^{kt}\right) = Ce^{kt}$$

$$P = \frac{Ce^{kt}}{1+\frac{1}{K}Ce^{kt}} \cdot \frac{\frac{1}{C}e^{-kt}}{\frac{1}{C}e^{-kt}}$$

$$P = \frac{1}{Ce^{-kt}+\frac{1}{K}} \cdot \frac{K}{K}$$

$$P = \frac{K}{Ae^{-kt}+1}$$

Solving for constant A

$P(0) = P_0$

$$P(0) = \frac{K}{A+1} = P_0$$

$$\Rightarrow K = P_0(A+1)$$

$$= P_0A + P_0$$

$$K - P_0 = P_0A$$

$$\Rightarrow A = \frac{K-P_0}{P_0}$$

$$\boxed{P(t) = \frac{K}{Ae^{-kt}+1} \text{ with } A = \frac{K-P_0}{P_0}}$$