

Quiz - Chapter 16 Solutions

① $\vec{F}(x, y, z) = \cos x \hat{i} + \sin y \hat{j} + e^{xy} \hat{k}$. Find $\text{curl } \vec{F}(1, 1, 1)$.

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & \sin y & e^{xy} \end{vmatrix} = \langle e^x - 0, -(e^y - 0), 0 - 0 \rangle = \langle e^x, -e^y, 0 \rangle$$

$$\text{curl } \vec{F}(x, y, z) = \langle e^x, -e^y, 0 \rangle \Rightarrow \text{curl } \vec{F}(1, 1, 1) = \langle e, -e, 0 \rangle = e\hat{i} - e\hat{j}$$

② Find $\int_C (y^2 - 3x^2) dx + (2xy + 2) dy$ with curve C from $(1, 1)$ to $(-1, 0)$.

$\vec{F} = \langle y^2 - 3x^2, 2xy + 2 \rangle$ conservative since $\frac{\partial Q}{\partial x} = 2y = \frac{\partial P}{\partial y}$.

Now find $f(x, y)$ such that $\vec{\nabla} f = \vec{F}$ (i.e., find the potential function.)

$$P = \frac{\partial f}{\partial x} = y^2 - 3x^2 \Rightarrow f(x, y) = \int y^2 - 3x^2 dx = xy^2 - x^3 + g(y)$$

$$Q = \frac{\partial f}{\partial y} = 2xy + 2 \Rightarrow f(x, y) = \int 2xy + 2 dy = xy^2 + 2y + h(x)$$

$$\text{So let } f(x, y) = xy^2 - x^3 + 2y$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= f|_{t=b} - f|_{t=a} = f(-1, 0) - f(1, 1) \text{ in this case.} \\ &= [(-1)(0^2) - (-1)^3 + 2(0)] - [1 \cdot 1^2 - 1^3 + 2(1)] \\ &= 1 - 2 = \boxed{-1} \end{aligned}$$

③ $\vec{F}(x, y, z) = (2xy + z^2)\hat{i} + x^2\hat{j} + (2xz + \pi \cos \pi z)\hat{k}$. Find the potential function.

Note: \vec{F} is conservative if there exists f such that $\vec{\nabla} f = \vec{F}$.

$$P = \frac{\partial f}{\partial x} = 2xy + z^2 \Rightarrow f(x, y, z) = \int 2xy + z^2 dx = x^2y + xz^2 + g(y, z)$$

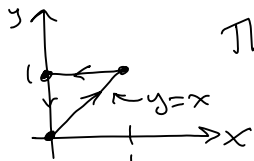
$$Q = \frac{\partial f}{\partial y} = x^2 \Rightarrow f(x, y, z) = \int x^2 dy = x^2y + h(x, z)$$

$$R = \frac{\partial f}{\partial z} = 2xz + \pi \cos \pi z \Rightarrow f(x, y, z) = \int 2xz + \pi \cos \pi z dz = x^2z + \sin \pi z + k(x, y)$$

$$\text{So let } f(x, y, z) = x^2y + xz^2 + \sin \pi z + C$$

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④ $C: (0,0) \rightarrow (1,1) \rightarrow (0,1) \rightarrow (0,0)$. Find $\int_C 2y dx - 3x dy$.



Use this for limits.

This is a Green's Theorem problem: Closed C .

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA \\ &= \iint_D (-3) - (2) dA \\ &= \int_0^1 \int_0^1 (-3-2) dy dx \\ Q &= \int_0^1 \int_0^y (-3-2) dx dy \end{aligned}$$

⑤ Work = $\int_C \vec{F} \cdot d\vec{r}$ $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \leq t \leq 2\pi$

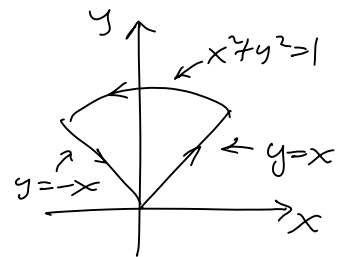
and $\vec{F} = \langle -zy, zx, xy \rangle$.

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_C \vec{F} \cdot \vec{r}'(t) dt \\ &= \int_0^{2\pi} \langle -zy, zx, xy \rangle \cdot \langle -\sin t, \cos t, 1 \rangle dt \\ &= \int_0^{2\pi} \langle -t \sin t, t \cos t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 1 \rangle dt \\ &= \int_0^{2\pi} t \sin^2 t + t \cos^2 t + \cos t \sin t dt \\ &= \int_0^{2\pi} t + \cos t \sin t dt \quad \begin{matrix} u = \sin t \\ du = \cos t dt \end{matrix} \\ &= \left. \frac{1}{2} t^2 + \frac{1}{2} \sin^2 t \right|_0^{2\pi} \\ &= \frac{1}{2} (2\pi)^2 + 0 - (0 + 0) \\ &= 2\pi^2 \end{aligned}$$

Bonus Use Green's Thm. to find $\int_C \vec{F} \cdot d\vec{r}$.
 $\vec{F} = \langle -16y + \sin x^2, 4e^y + 3x^2 \rangle$ $C \rightarrow$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA \\ &= \iint_D (6x - 16) dA \end{aligned}$$

use polar
 $r: 0 \rightarrow 1$
 $\theta: \frac{\pi}{4} \rightarrow \frac{3\pi}{4}$



$$\begin{aligned} &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^1 (6r \cos \theta + 16) r dr d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2r^3 \cos \theta + 8r^2 \Big|_0^1 d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2 \cos \theta + 8 d\theta = 2 \sin \theta + 8\theta \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\ &= 2 \left(\frac{\sqrt{2}}{2} \right) + 8 \left(\frac{3\pi}{4} \right) - 2 \left(\frac{\sqrt{2}}{2} \right) - 8 \left(\frac{\pi}{4} \right) = 6\pi - 2\pi = 4\pi \end{aligned}$$

-Abby Brown 5/13/04