Dot Product Formula with $\cos\theta$

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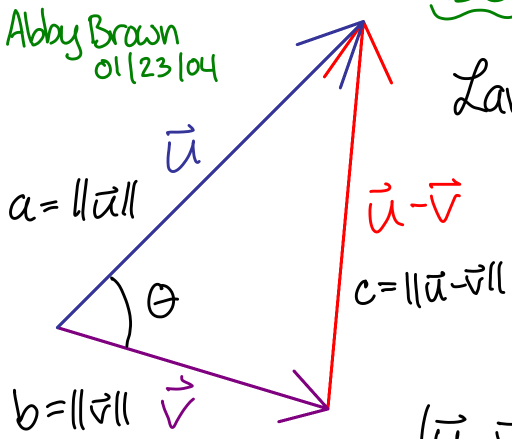


Law of Cosines: $c^2=a^2+b^2-2ab\cos\theta$

$$\|\vec{u}-\vec{v}\|^2=\|\vec{u}\|^2+\|\vec{v}\|^2-2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

From definition of dot product and pyth. thm $\Rightarrow \|\vec{u}\|^2=\vec{u}\cdot\vec{u}$

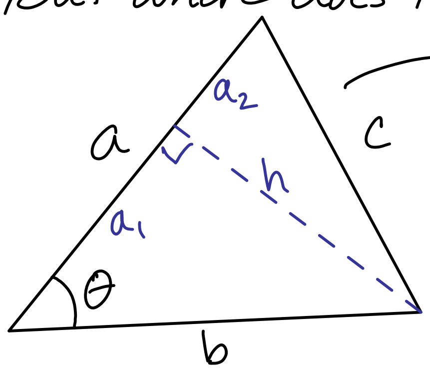
$$(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})=\vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{v}-2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\cancel{\vec{u}\cdot\vec{u}}-\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}+\cancel{\vec{v}\cdot\vec{v}}=\cancel{\vec{u}\cdot\vec{u}}+\cancel{\vec{v}\cdot\vec{v}}-2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$-2\vec{u}\cdot\vec{v}=-2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\boxed{\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\theta}$$

But where does the Law of Cosines come from?



$a_2^2+h^2=c^2$ and $a_1^2+h^2=b^2$

$h^2=c^2-a_2^2$ $\quad$ $h^2=b^2-a_1^2$

$$c^2-a_2^2=b^2-a_1^2$$

$$c^2=b^2-a_1^2+a_2^2$$

$$c^2=b^2-a_1^2-a_1^2+a_1^2+a_2^2$$

$$c^2=b^2-2a_1^2+a^2-2a_1a_2$$

rearrange

$$c^2=a^2+b^2-2a_1^2-2a_1a_2$$

$$c^2=a^2+b^2-2a_1(a_1+a_2)$$

$$c^2=a^2+b^2-2b\cos\theta(a)$$

$$\boxed{c^2=a^2+b^2-2ab\cos\theta}$$

Hey! It worked!

Also, $a=a_1+a_2$

$a^2=a_1^2+2a_1a_2+a_2^2$

$\underline{a_1^2+a_2^2}=a^2-2a_1a_2$

From triangle $\cos\theta=\dfrac{a_1}{b}$

$\Rightarrow a_1=b\cos\theta$

There may be a more efficient way, but this is what I came up with.

What about the formula with sine: $\|\vec{u}\times\vec{v}\|=\|\vec{u}\|\|\vec{v}\|\sin\theta$?

$\Rightarrow$ I went through this in the notes, but I really just used this formula to explain $\|\vec{u}\times\vec{v}\|$ = area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.

The $\|\vec{u}\times\vec{v}\|=\|\vec{u}\|\|\vec{v}\|\sin\theta$ formula can also be proved algebraically. The proof includes the dot product formula above. (See p. 806 in book.)