6.5 Notes ⇒ Change of Basis Problem

The "Change of Basis Problem" is the focus for section 6.5, although there is other key information in this section too.

First let's start with a simple example — actually it's a problem we have solved before. (6.5 #6a)

⇒ Find the coordinate matrix for $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ relative to the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$.

This means we want the coefficients for the $\vec{v}_i$'s in the linear combination to construct $\vec{v}$.

$$\vec{v} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$$

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + c\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \leftarrow \text{Find } a, b, \text{ and } c.$$

This gives us a system that, of course, can be written as a matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 3 & 3 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{c|c} I & \begin{matrix} 3 \\ -2 \\ 1 \end{matrix} \end{array}\right]$$

Side Question: How do we know the left side will RREF to I?

So, $a = 3$, $b = -2$, $c = 1$

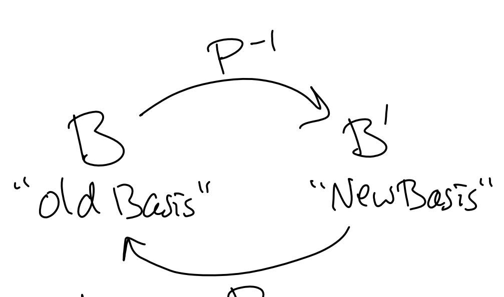
$$[\vec{v}]_S = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Now the problem gets more complicated for two reasons.

① We want to find an easy way to change any vector to the new basis.

② Neither basis needs to be the standard basis.

Here's the idea:



"old" & "new" → are poor choices since we can go either way.

Where P is the "transition matrix" to convert a vector in B' to B.

$$[\vec{v}]_B = P[\vec{v}]_{B'}$$

↖ matrix multiplication

$$\text{So } [\vec{v}]_{B'} = P^{-1}[\vec{v}]_B$$

— continued —

Once we know P (and P^{-1}) we can convert any vector in either basis to the other basis. So now we just need P.

The matrix P has as its column vectors each basis vector of B' written in terms of the basis we're converting to: B. This means that if we know how to convert each basis vector, then those conversions just get applied (via matrix multiplication) to vectors in the space and they will be converted the same way. (In other words, if you can convert the basis vectors then you can convert any vector built with that basis.)

$$P = \left[[\vec{u}_1']_B \,\middle|\, [\vec{u}_2']_B \,\middle|\, \cdots \,\middle|\, [\vec{u}_n']_B \right]$$

← each column is a B' basis vector converted to basis B.

Let's work though #13 and I'll try to explain along the way.

$B = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, $B' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ in $\mathbb{R}^3$

$\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

⇒ Find transition matrix from B to B'.

So we need to convert each basis vector in B to the basis we want to change to (B').

$$P = \left[\cdots \,\middle|\, [\text{original Basis vector}]_{\text{Basis we are changing to}} \,\middle|\, \cdots \right]$$

← 1st B basis vector. ← to find coordinates relative to B' basis vectors.

$\vec{u}_1 = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ ← Just like problem before (#6a)

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = a\begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \begin{array}{l} 3a + 1b + -1c = 2 \\ 1a + 1b + 0c = 1 \\ -5a + 3b + 2c = 1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 3 & 1 & -1 & 2 \\ 1 & 1 & 0 & 1 \\ -5 & -3 & 2 & 1 \end{array}\right]$$

$$\text{RREF} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow a = 3, b = -2, c = 5 \Rightarrow [\vec{u}_1]_{B'} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

This is the first column of the transition matrix P. ↗

—continued—

But we also have to do this for $\vec{u}_2$ and $\vec{u}_3$.

$\vec{u}_2 = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$

$$\begin{bmatrix}2\\-1\\1\end{bmatrix} = a\begin{bmatrix}3\\1\\-5\end{bmatrix} + b\begin{bmatrix}1\\1\\-3\end{bmatrix} + c\begin{bmatrix}-1\\0\\2\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}3 & 1 & -1 & 2\\1 & 1 & 0 & -1\\-5 & -3 & 2 & 1\end{array}\right]$$

RREF $\Rightarrow a=2, b=-3, c=1$

$\vec{u}_3 = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$

$$\begin{bmatrix}1\\2\\1\end{bmatrix} = a\begin{bmatrix}3\\1\\-5\end{bmatrix} + b\begin{bmatrix}1\\1\\-3\end{bmatrix} + c\begin{bmatrix}-1\\0\\2\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}3 & 1 & -1 & 1\\1 & 1 & 0 & 2\\-5 & -3 & 2 & 1\end{array}\right]$$

RREF $\Rightarrow a = 5/2$, $b = -\frac{1}{2}$, $c=6$

However, notice that all three matrices are the same except for the $\vec{b}$ vector (as in $A\vec{x}=\vec{b}$). So instead of RREF-ing three times, we can augment the first matrix with the other two $\vec{b}$ vectors and RREF just once.

$$\left[\begin{array}{ccc|c|c|c}3 & 1 & -1 & 2 & 2 & 1\\1 & 1 & 0 & 1 & -1 & 2\\-5 & -3 & 2 & 1 & 1 & 1\end{array}\right]$$

← This is just [B′ vectors | B vectors] (B′: Basis we are changing to; B: Original Basis)

⇓ RREF

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 3 & 2 & 5/2\\0 & 1 & 0 & -2 & -3 & -1/2\\0 & 0 & 1 & 5 & 1 & 6\end{array}\right]$$

← And it's already in matrix form.

$$P = \begin{bmatrix}3 & 2 & 5/2\\-2 & -3 & -1/2\\5 & 1 & 6\end{bmatrix}$$

So, if $[\vec{w}]_B$ is a vector in B then $P[\vec{w}]_B$ will write this vector in B′. (↑ matrix multiplication)

Example: $\vec{w} = \begin{bmatrix}-5\\8\\-5\end{bmatrix}$ is in $\mathbb{R}^3$. First convert it to B.

$$\vec{w} = a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 \Rightarrow \begin{bmatrix}-5\\8\\-5\end{bmatrix} = a\begin{bmatrix}2\\1\\1\end{bmatrix} + b\begin{bmatrix}2\\-1\\1\end{bmatrix} + c\begin{bmatrix}1\\2\\1\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}2 & 2 & 1 & -5\\1 & -1 & 2 & 8\\1 & 1 & 1 & -5\end{array}\right]$$

$a=9, b=-9, c=-5$

$$\therefore [\vec{w}]_B = \begin{bmatrix}9\\-9\\-5\end{bmatrix}$$

Using our transition matrix, $[\vec{w}]_{B'} = P[\vec{w}]_B = P\begin{bmatrix}9\\-9\\-5\end{bmatrix}$

$$[\vec{w}]_{B'} = \begin{bmatrix}3 & 2 & 5/2\\-2 & -3 & -1/2\\5 & 1 & 6\end{bmatrix}\begin{bmatrix}9\\-9\\-5\end{bmatrix} = \begin{bmatrix}-7/2\\23/2\\6\end{bmatrix}$$

Check: Does $\vec{w} = -\frac{7}{2}\vec{v}_1 + \frac{23}{2}\vec{v}_2 + 6\vec{v}_3$? ← Linear combo. of B′ vectors with coefficients from $[\vec{w}]_{B'}$

$$\begin{bmatrix}-5\\8\\-5\end{bmatrix} \stackrel{?}{=} -\frac{7}{2}\begin{bmatrix}3\\1\\-5\end{bmatrix} + \frac{23}{2}\begin{bmatrix}1\\1\\-3\end{bmatrix} + 6\begin{bmatrix}-1\\0\\2\end{bmatrix}$$

OR to find $[\vec{w}]_{B'}$ directly, solve $\begin{bmatrix}-5\\8\\-5\end{bmatrix} = a\begin{bmatrix}3\\1\\-5\end{bmatrix} + b\begin{bmatrix}1\\1\\-3\end{bmatrix} + c\begin{bmatrix}-1\\0\\2\end{bmatrix}$.

—Abby Brown 5/12/04