Orthogonal Diagonalization Example

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

First we need to find the eigenvalues and eigenvectors.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 2 & -4 \\ 2 & \lambda-6 & -2 \\ -4 & -2 & \lambda-3 \end{vmatrix} = (\lambda-3)\big(\overbrace{(\lambda-6)(\lambda-3)-4}^{\lambda^2-9\lambda+18-4}\big) - 2\big(2(\lambda-3)-8\big) - 4\big(-4+4(\lambda-6)\big)$$

$$= \lambda^3 - 9\lambda^2 + 18\lambda - 4\lambda - 3\lambda^2 + 27\lambda - 54 + 12 - 4\lambda + 12 + 16 + 16 - 16\lambda + 96$$

$$= \lambda^3 - 12\lambda^2 + 21\lambda + 98 = (\lambda+2)(\lambda-7)^2$$

Note: You may use technology to help with the algebra for longer polynomials. Be sure you know how to set it all up.

Characteristic Equation: $(\lambda+2)(\lambda-7)^2 = 0$

$\Rightarrow \lambda = -2$ (with algebraic multiplicity 1)

$\lambda = 7$ (with algebraic multiplicity 2)

$\lambda = -2$

$$(\lambda I - A)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} -5 & 2 & -4 & 0 \\ 2 & -8 & -2 & 0 \\ -4 & -2 & -5 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -t \\ x_2 = -\frac{1}{2}t \\ x_3 = t \end{array} \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} t$$

$\lambda = 7$

$$(\lambda I - A)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 4 & 2 & -4 & 0 \\ 2 & 1 & -2 & 0 \\ -4 & -2 & 4 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = t - \frac{1}{2}s \\ x_2 = s \\ x_3 = t \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} s$$

Eigenspace Basis for $\lambda = -2$: $\left\{ \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$ ← Geometric Multiplicity 1

Eigenspace Basis for $\lambda = 7$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$ ← Geometric Multiplicity 2

A is <u>diagonalizable</u> since the algebraic multiplicity equals the geometric multiplicity for <u>each</u> eigenvalue.

(continued)

Want P such that $P^{-1}AP = D$ where D is diagonal.

$P = \left[\vec{p}_1 \,|\, \vec{p}_2 \,|\, \vec{p}_3\right]$ where the $\vec{p}_i$ are the eigenvectors of A.

$$\boxed{P = \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}}$$

Note: Position of λs correspond to positions of eigenvectors.

Check: $P^{-1}AP = \frac{1}{9}\begin{bmatrix} -4 & -2 & 4 \\ 4 & 2 & 5 \\ -2 & 8 & 2 \end{bmatrix}\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = D$

P^{-1}

Note: Eigenvectors may be scalar multiples of eigenvectors.

$\vec{x} = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} t \rightarrow$ let $t = -2$ $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}$, $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} s \rightarrow$ let $t=1$, $s=2$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

Then $P = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix}$ and $P^{-1}AP = \frac{1}{9}\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 5 \\ 1 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = D$

$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ is <u>orthogonally diagonalizable</u> since A is <u>symmetric</u>.

For orthogonal diagonalization, P must be an orthogonal matrix. $P^{-1} = P^T \Rightarrow P^TAP = D$. The columns of P must form an <u>orthonormal</u> set. We need the Gram-Schmidt process.

A is symmetric $\Rightarrow$ eigenvalues from different eigenspaces are orthogonal.

$\lambda = -2: \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}$ $\quad \lambda = 7: \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

$\uparrow \vec{u}_1 \qquad \uparrow \vec{u}_2 \qquad \uparrow \vec{u}_3$

$\vec{u}_1 \cdot \vec{u}_2 = 2 + 0 + -2 = 0$

$\vec{u}_1 \cdot \vec{u}_3 = -2 + 2 + 0 = 0$

(continued)

Since different eigenspace vectors are already orthogonal, we do not need the Gram-Schmidt process on them all together. We only need to appy it to each separate set.

$$\left\{\begin{bmatrix}2\\1\\-2\end{bmatrix}\right\} \Rightarrow \vec{V}_1 = \begin{bmatrix}2\\1\\-2\end{bmatrix} \Rightarrow \vec{P}_1 = \frac{\vec{V}_1}{\|\vec{V}_1\|} = \begin{bmatrix}2/3\\1/3\\-2/3\end{bmatrix}$$

$$\|\vec{V}_1\| = \sqrt{4+1+4} = 3$$

← Gram-Schmidt on one vector just normalizes it.

$$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix} \begin{bmatrix}-1\\2\\0\end{bmatrix}\right\} \Rightarrow \vec{V}_2 = \vec{u}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$\uparrow \vec{u}_2$ $\uparrow \vec{u}_3$

$$\|\vec{V}_2\| = \sqrt{1+0+1} = \sqrt{2}$$

$$\vec{V}_3 = \vec{u}_3 - \operatorname{proj}_{\vec{V}_2} \vec{u}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{V}_2}{\|\vec{V}_2\|^2} \vec{V}_2$$

Only applying Gram-Schmidt to two vectors.

$$= \begin{bmatrix}-1\\2\\0\end{bmatrix} - \frac{-1}{(\sqrt{2})^2} \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}-1+\frac{1}{2}\\2+0\\0+\frac{1}{2}\end{bmatrix} = \begin{bmatrix}-1/2\\2\\1/2\end{bmatrix} = \vec{V}_3$$

$$\|\vec{V}_3\| = \sqrt{\tfrac{1}{4}+4+\tfrac{1}{4}} = \tfrac{3}{2}\sqrt{2}$$

$$\vec{P}_2 = \frac{\vec{V}_2}{\|\vec{V}_2\|} = \begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}, \quad \vec{P}_3 = \frac{\vec{V}_3}{\|\vec{V}_3\|} = \begin{bmatrix}\frac{-1}{3\sqrt{2}}\\\frac{4}{3\sqrt{2}}\\\frac{1}{3\sqrt{2}}\end{bmatrix}$$

$$\boxed{P = \begin{bmatrix}2/3 & \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}}\\ 1/3 & 0 & \frac{4}{3\sqrt{2}}\\ -2/3 & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}}\end{bmatrix}}$$

$\uparrow \vec{P}_1$ $\uparrow \vec{P}_2$ $\uparrow \vec{P}_3$

$$P^{-1} = \begin{bmatrix}\frac{2}{3} & 1/3 & \frac{-2}{3}\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{-1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}}\end{bmatrix} = P^T$$

And $P^T A P = \begin{bmatrix}-2 & 0 & 0\\0 & 7 & 0\\0 & 0 & 7\end{bmatrix} = D$

All symmetric matrices are orthogonally diagonalizable. However, even if the entries of the matrix are "nice" that does not mean the arithmetic will work out nicely, especially through the Gram-Schmidt process. I found this matrix in an example online and thought that it worked out reasonably well so I used it to develop my example here. https://media.ed.science.psu.edu/sites/media/ed/files/video/5.3v6example3.mp4

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