Quiz - Chapter 16 Solutions

① $\vec{F}(x,y,z) = \cos x\,\hat{i} + \sin y\,\hat{j} + e^{xy}\,\hat{k}$. Find $\text{curl}\,\vec{F}(1,1,1)$.

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & \sin y & e^{xy} \end{vmatrix} = \langle xe^{xy} - 0, -(ye^{xy} - 0), 0 - 0 \rangle = \langle xe^{xy}, -ye^{xy}, 0 \rangle$$

$$\text{curl}\,\vec{F}(x,y,z) = \langle xe^{xy}, -ye^{xy}, 0\rangle \Rightarrow \text{curl}\,\vec{F}(1,1,1) = \langle e, -e, 0\rangle = e\hat{i} - e\hat{j}$$

② Find $\int_C (y^2 - 3x^2)\,dx + (2xy + 2)\,dy$ with curve C from $(1,1)$ to $(-1,0)$.

$\vec{F} = \langle y^2 - 3x^2, 2xy + 2\rangle$ conservative since $\frac{\partial Q}{\partial x} = 2y = \frac{\partial P}{\partial y}$.

Now find $f(x,y)$ such that $\vec{\nabla} f = \vec{F}$ (i.e., find the potential function.)

$$P = \frac{\partial f}{\partial x} = y^2 - 3x^2 \Rightarrow f(x,y) = \int y^2 - 3x^2\,dx = xy^2 - x^3 + g(y)$$

$$Q = \frac{\partial f}{\partial y} = 2xy + 2 \Rightarrow f(x,y) = \int 2xy + 2\,dy = xy^2 + 2y + h(x)$$

So let $f(x,y) = xy^2 - x^3 + 2y$

$$\int_C \vec{F}\cdot d\vec{r} = f\Big|_{t=b} - f\Big|_{t=a} = f(-1,0) - f(1,1) \text{ in this case.}$$

$$= \left[(-1)(0^2) - (-1)^3 + 2(0)\right] - \left[1\cdot 1^2 - 1^3 + 2(1)\right]$$

$$= 1 - 2 = -1$$

③ $\vec{F}(x,y,z) = (2xy + z^2)\hat{i} + x^2\hat{j} + (2xz + \pi\cos\pi z)\hat{k}$. Find the potential function.

Note: $\vec{F}$ is conservative if there exists f such that $\vec{\nabla} f = \vec{F}$.

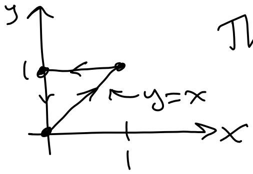
$$P = \frac{\partial f}{\partial x} = 2xy + z^2 \Rightarrow f(x,y,z) = \int 2xy + z^2\,dx = x^2y + xz^2 + g(y,z)$$

$$Q = \frac{\partial f}{\partial y} = x^2 \Rightarrow f(x,y,z) = \int x^2\,dy = x^2y + h(x,z)$$

$$R = \frac{\partial f}{\partial z} = 2xz + \pi\cos\pi z \Rightarrow f(x,y,z) = \int 2xz + \pi\cos\pi z\,dz = x^2z + \sin\pi z + k(x,y)$$

So let $f(x,y,z) = x^2y + xz^2 + \sin\pi z + C$

(4) C: $(0,0) \to (1,1) \to (0,1) \to (0,0)$. Find $\int_C \overset{P}{2y}\,dx - \overset{Q}{3x}\,dy$.



Use this for limits.

This is a Green's Theorem problem : Closed C.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$
$$= \iint_D (-3) - (2)\,dA$$
$$= \int_0^1\int_x^1 (-3-2)\,dy\,dx$$
$$\text{OR } = \int_0^1\int_0^y (-3-2)\,dx\,dy$$

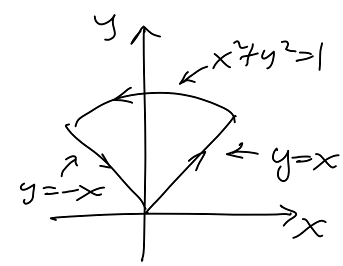
(5) Work $= \int_C \vec{F}\cdot d\vec{r}$ $\quad \vec{r}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 2\pi$

and $\vec{F} = \langle -zy, zx, xy\rangle$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot \vec{r}\,'(t)\,dt$$
$$= \int_0^{2\pi} \langle -zy, zx, xy\rangle \cdot \langle -\sin t, \cos t, 1\rangle\,dt$$
$$= \int_0^{2\pi} \langle -t\sin t, t\cos t, \cos t\sin t\rangle \cdot \langle -\sin t, \cos t, 1\rangle\,dt$$
$$= \int_0^{2\pi} t\sin^2 t + t\cos^2 t + \cos t\sin t\,dt$$
$$= \int_0^{2\pi} t + \cos t\sin t\,dt \qquad \begin{matrix} u = \sin t \\ du = \cos t\,dt \end{matrix}$$
$$= \frac{1}{2}t^2 + \frac{1}{2}\sin^2 t \Big|_0^{2\pi}$$
$$= \frac{1}{2}(2\pi)^2 + 0 - (0+0)$$
$$= 2\pi^2$$

(Bonus) Use Green's Thm. to find $\int_C \vec{F}\cdot d\vec{r}$.

$\vec{F} = \langle \underbrace{-16y + \sin x^2}_{P}, \underbrace{4e^y + 3x^2}_{Q}\rangle$ C →



$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$
$$= \iint_D 6x - {-16}\,dA$$

use polar

$r: 0 \to 1$

$\theta: \frac{\pi}{4} \to \frac{3\pi}{4}$

$$= \int_{\pi/4}^{3\pi/4}\int_0^1 (6r\cos\theta + 16)\,r\,dr\,d\theta$$
$$= \int_{\pi/4}^{3\pi/4} 2r^3\cos\theta + 8r^2 \Big|_0^1\,d\theta$$
$$= \int_{\pi/4}^{3\pi/4} 2\cos\theta + 8\,d\theta = 2\sin\theta + 8\theta \Big|_{\pi/4}^{3\pi/4}$$
$$= 2\left(\tfrac{\sqrt{2}}{2}\right) + 8\left(\tfrac{3\pi}{4}\right) - 2\left(\tfrac{\sqrt{2}}{2}\right) - 8\left(\tfrac{\pi}{4}\right) = 6\pi - 2\pi = 4\pi$$

– Abby Brown 5/13/04