Eigenvalues & Eigenvectors

$A\vec{x} = \lambda\vec{x}$ λ is a scalar

$\lambda\vec{x} - A\vec{x} = \vec{0}$

$\lambda I\vec{x} - A\vec{x} = \vec{0}$

$(\lambda I - A)\vec{x} = \vec{0}$ ← want to find nontrivial solns.

$\to B\vec{x} = \vec{0}$

If $B\vec{x} = \vec{0}$ has only the trivial soln, then B is invertible

we need $B\vec{x} = \vec{0}$ to have nontrivial soln, B must be singular

Want B to be singular

"B" $= \lambda I - A$

A is inv. $\Leftrightarrow \det(A) \neq 0$

A is singular $\Leftrightarrow \det(A) = 0$

$\det(\lambda I - A) = 0$ ← characteristic equation

Ex: $A = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \lambda-2 & -3 \\ -4 & \lambda-3 \end{bmatrix}$

$\begin{vmatrix} \lambda-2 & -3 \\ -4 & \lambda-3 \end{vmatrix} = (\lambda-2)(\lambda-3) - 12$

$= \lambda^2 - 5\lambda + 6 - 12$

$= \lambda^2 - 5\lambda - 6$ ← want $\det(\lambda I - A) = 0$

$= (\lambda - 6)(\lambda + 1) = 0$

$\lambda = 6, \lambda = -1$ ← eigenvalues

$(\lambda I - A)\vec{x} = \vec{0}$ ← solve this system for each λ

$\lambda = 6$ $\lambda I - A = \begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 4 & -3 & 0 \\ -4 & 3 & 0 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 4 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 = \frac{3t}{4}$

$x_2 = t$

$\Rightarrow \vec{x} = \begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix}$ ← eigenvector for $\lambda = 6$

$\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$A\vec{x} = \lambda\vec{x}$

Ex: $\vec{x} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 36 \\ 48 \end{bmatrix} = 6\begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$A\ \vec{x} = \lambda\vec{x}$

$\lambda = -1$ $\left[\begin{array}{cc|c} -3 & -3 & 0 \\ -4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 = -t$

$x_2 = t$

$\vec{x} = \begin{bmatrix} -t \\ t \end{bmatrix}$

$\begin{bmatrix} t \\ -t \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Ex: $\vec{x} = \begin{bmatrix} -7 \\ 7 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -7 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \end{bmatrix} = (-1)\begin{bmatrix} -7 \\ 7 \end{bmatrix}$

$A\ \vec{x} = \lambda\vec{x}$